Lecture Notes: Spin-1/2 Ensemble

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Tohoku University, June 2025

Contents

1 Introduction to Group Theory and Representations

Group theory provides a mathematical framework to study symmetries in physical systems, especially in quantum mechanics.

1.1 Basic Definitions

A **group** G is a set with a binary operation satisfying:

- **Closure**: $a, b \in G \Rightarrow ab \in G$
- **Associativity**: $(ab)c = a(bc)$
- **Identity**: There exists $e \in G$ such that $ae = ea = a$
- **Inverse**: For each $a \in G$, there exists $a^{-1} \in G$ with $aa^{-1} = e$

1.2 Lie Groups: $\mathrm{SU}(2)$ and $\mathrm{SO}(3)$

- $\mathrm{SU}(2)$: Set of 2×2 unitary matrices with determinant 1. Describes spin-1/2 systems.
- $\mathrm{SO}(3)$: 3×3 real orthogonal matrices with determinant 1. Describes spatial rotations.

$\mathrm{SU}(2)$ is the double cover of $\mathrm{SO}(3)$, meaning each $R \in \mathrm{SO}(3)$ corresponds to two elements in $\mathrm{SU}(2)$.

1.3 Clebsch-Gordan Decomposition

Combining angular momenta j_1 and j_2:

$$j_1 \otimes j_2 = |j_1 - j_2| \oplus \cdots \oplus (j_1 + j_2)$$

2 Spin-1/2 Systems

2.1 Hilbert Space and Basis

The state space for a spin-1/2 particle is two-dimensional:

$$\mathscr{H}_{1/2} = \mathrm{span}\{|\uparrow\rangle, |\downarrow\rangle\}$$

2.2 Spin Operators and Pauli Matrices

$$\hat{S}_i = \frac{\hbar}{2}\sigma_i \quad (i = x, y, z)$$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

2.3 Eigenstates of $\hat{S}_z$

$$\hat{S}_z|\uparrow\rangle = \frac{\hbar}{2}|\uparrow\rangle, \quad \hat{S}_z|\downarrow\rangle = -\frac{\hbar}{2}|\downarrow\rangle$$

3 Two Spin-1/2 Systems

3.1 Tensor Product Basis

$$\mathscr{H} = \mathscr{H}_{1/2} \otimes \mathscr{H}_{1/2}$$

Basis:

$$\{|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle\}$$

3.2 Total Spin Operator

$$\hat{\mathbf{J}} = \hat{\mathbf{S}}_1 + \hat{\mathbf{S}}_2$$

3.3 Clebsch-Gordan Basis

- Triplet states ($j = 1$):

$$|1,1\rangle = |\uparrow\uparrow\rangle \tag{1}$$

$$|1,0\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) \tag{2}$$

$$|1,-1\rangle = |\downarrow\downarrow\rangle \tag{3}$$

- Singlet state ($j = 0$):

$$|0,0\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$$

4 N Spin-1/2 Systems

4.1 Hilbert Space Dimension

$$\dim(\mathscr{H}) = 2^N$$

4.2 Symmetric Subspace

The totally symmetric states span a subspace of dimension $N + 1$.

4.3 Angular Momentum Decomposition

Using recursive Clebsch-Gordan decomposition:

$$\left(\frac{1}{2}\right)^{\otimes N} = \bigoplus_j d_j \cdot \mathscr{H}_j$$

where d_j is the multiplicity of spin-j irrep.

4.4 Dicke States

$$|D(N,m)\rangle = \text{symmetrized combination of } m \text{ spin-ups and } N - m \text{ spin-downs}$$

5 Dimension Reduction via Clebsch-Gordan and Dicke Basis

5.1 Motivation

The full space 2^N grows exponentially. Group symmetry helps reduce to a manageable form.

5.2 Clebsch-Gordan Trees

Build total angular momentum via repeated addition of spin-1/2 particles.

5.3 Collective Angular Momentum Operators

$$\hat{J}_z|j,m\rangle = \hbar m|j,m\rangle, \quad \hat{J}^2|j,m\rangle = \hbar^2 j(j+1)|j,m\rangle$$

5.4 Example: Three Spins

$$\frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1$$
$$1 \otimes \frac{1}{2} = \frac{1}{2} \oplus \frac{3}{2}$$

Total decomposition:

$$\left(\frac{1}{2}\right)^{\otimes 3} = \frac{1}{2}^{\oplus 2} \oplus \frac{3}{2}$$

5.5 Summary

We presented a group-theoretic view of spin systems, introducing SU(2), Clebsch-Gordan decomposition, and the Dicke basis to systematically reduce the Hilbert space of many-spin systems.